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THEORY GUIDE

Equations of Fluid Flow

Kinetic Energy Equation in Cartesian and Cylindrical Coordinates

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1 Kinetic energy equation in Cartesian coordinates

In Ref. [1] we derived the continuity equation for a three-dimensional unsteady compressible fluid flow in Cartesian coordinates:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \quad [\mathrm{kg\ m^{-3}\ s^{-1}}] \quad (1.1)$$

In Ref. [2] we derived the equations for the three components of momentum in a three-dimensional unsteady compressible fluid flow in Cartesian coordinates. When written in terms of stress, the equations are

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z} = -\frac{\partial p}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \rho f_x \quad [\mathrm{kg\ m^{-2}\ s^{-2}}] \quad (1.2)$$

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho vu)}{\partial x} + \frac{\partial(\rho v^2)}{\partial y} + \frac{\partial(\rho vw)}{\partial z} = -\frac{\partial p}{\partial y} + \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + \rho f_y \quad [\mathrm{kg\ m^{-2}\ s^{-2}}] \quad (1.3)$$

$$\frac{\partial(\rho w)}{\partial t} + \frac{\partial(\rho wu)}{\partial x} + \frac{\partial(\rho wv)}{\partial y} + \frac{\partial(\rho w^2)}{\partial z} = -\frac{\partial p}{\partial z} + \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho f_z \quad [\mathrm{kg\ m^{-2}\ s^{-2}}] \quad (1.4)$$

The kinetic energy per unit mass of the fluid flow K is $(u^2 + v^2 + w^2)/2$. We can derive an equation for K from the continuity equation and the three momentum equations. The left-hand side of the x-component momentum equation (1.2) can be expanded to give

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z} + u\frac{\partial \rho}{\partial t} + u\frac{\partial(\rho u)}{\partial x} + u\frac{\partial(\rho v)}{\partial y} + u\frac{\partial(\rho w)}{\partial z}$$

$$= -\frac{\partial p}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \rho f_x$$

From the continuity equation (1.1) the last four terms on the left-hand side of this equation are zero, so

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \rho f_x$$

By multiplying through by u, we have

$$\rho\frac{\partial\left(\frac{u^2}{2}\right)}{\partial t} + \rho u\frac{\partial\left(\frac{u^2}{2}\right)}{\partial x} + \rho v\frac{\partial\left(\frac{u^2}{2}\right)}{\partial y} + \rho w\frac{\partial\left(\frac{u^2}{2}\right)}{\partial z} = -u\frac{\partial p}{\partial x} + u\left[\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}\right] + \rho u f_x \quad (1.5)$$

Similarly, for the y- and z-component momentum equations:

$$\rho\frac{\partial\left(\frac{v^2}{2}\right)}{\partial t}+\rho u\frac{\partial\left(\frac{v^2}{2}\right)}{\partial x}+\rho v\frac{\partial\left(\frac{v^2}{2}\right)}{\partial y}+\rho w\frac{\partial\left(\frac{v^2}{2}\right)}{\partial z}=-v\frac{\partial p}{\partial y}+v\left[\frac{\partial\sigma_{yx}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{yz}}{\partial z}\right]+\rho v f_y \quad (1.6)$$

$$\rho\frac{\partial\left(\frac{w^2}{2}\right)}{\partial t}+\rho u\frac{\partial\left(\frac{w^2}{2}\right)}{\partial x}+\rho v\frac{\partial\left(\frac{w^2}{2}\right)}{\partial y}+\rho w\frac{\partial\left(\frac{w^2}{2}\right)}{\partial z}=-w\frac{\partial p}{\partial z}+w\left[\frac{\partial\sigma_{zx}}{\partial x}+\frac{\partial\sigma_{zy}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}\right]+\rho w f_z \quad (1.7)$$

By summing (1.5), (1.6) and (1.7), we obtain

$$\rho\frac{\partial K}{\partial t}+\rho u\frac{\partial K}{\partial x}+\rho v\frac{\partial K}{\partial y}+\rho w\frac{\partial K}{\partial z}$$

$$=-\left[u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+w\frac{\partial p}{\partial z}\right]$$

$$+u\left[\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\sigma_{xz}}{\partial z}\right]+v\left[\frac{\partial\sigma_{yx}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{yz}}{\partial z}\right]+w\left[\frac{\partial\sigma_{zx}}{\partial x}+\frac{\partial\sigma_{zy}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}\right]$$

$$\rho u f_x+\rho v f_y+\rho w f_z \quad [\mathrm{J\ m^{-3}\ s^{-1}}] \quad (1.8)$$

Multiplying the continuity equation (1.1) by K gives

$$K\frac{\partial\rho}{\partial t}+K\frac{\partial(\rho u)}{\partial x}+K\frac{\partial(\rho v)}{\partial y}+K\frac{\partial(\rho w)}{\partial z}=0$$

and adding this equation to the left-hand side of (1.8) gives

$$\frac{\partial(\rho K)}{\partial t}+\frac{\partial(\rho u K)}{\partial x}+\frac{\partial(\rho v K)}{\partial y}+\frac{\partial(\rho w K)}{\partial z}$$

$$=-\left[u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+w\frac{\partial p}{\partial z}\right]$$

$$+u\left[\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\sigma_{xz}}{\partial z}\right]+v\left[\frac{\partial\sigma_{yx}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{yz}}{\partial z}\right]+w\left[\frac{\partial\sigma_{zx}}{\partial x}+\frac{\partial\sigma_{zy}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}\right]$$

$$\rho u f_x+\rho v f_y+\rho w f_z \quad [\mathrm{J\ m^{-3}\ s^{-1}}] \quad (1.9)$$

which is the equation for the conservation of kinetic energy K in Cartesian coordinates.

2 Kinetic energy equation in cylindrical coordinates

In Ref. [1] we derived the continuity equation for a three-dimensional unsteady compressible fluid flow in cylindrical coordinates:

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial (r\rho v_r)}{\partial r}+\frac{1}{r}\frac{\partial (\rho v_\theta)}{\partial \theta}+\frac{\partial (\rho v_z)}{\partial z}=0 \quad [\mathrm{kg\ m^{-3}\ s^{-1}}] \quad (2.1)$$

In Ref. [3] we derived the equations for the three components of momentum in a three-dimensional unsteady compressible fluid flow in cylindrical coordinates. When written in terms of stress, the equations are

$$\frac{\partial (\rho v_r)}{\partial t}+\frac{1}{r}\frac{\partial (r\rho {v_r}^2)}{\partial r}+\frac{1}{r}\frac{\partial (\rho v_r v_\theta)}{\partial \theta}+\frac{\partial (\rho v_r v_z)}{\partial z}$$

$$=-\frac{\partial p}{\partial r}+\frac{\partial \sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta}+\frac{\partial \sigma_{rz}}{\partial z}+\frac{\sigma_{rr}}{r}-\frac{\sigma_{\theta\theta}}{r}+\rho f_r \quad (2.2)$$

$$\frac{\partial (\rho v_\theta)}{\partial t}+\frac{1}{r}\frac{\partial (r\rho v_r v_\theta)}{\partial r}+\frac{1}{r}\frac{\partial (\rho {v_\theta}^2)}{\partial \theta}+\frac{\partial (\rho v_\theta v_z)}{\partial z}$$

$$=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta}+\frac{\sigma_{\theta r}}{r}+\frac{\partial \sigma_{\theta r}}{\partial r}+\frac{\partial \sigma_{\theta z}}{\partial z}+\frac{\sigma_{r\theta}}{r}+\rho f_\theta \quad (2.3)$$

$$\frac{\partial (\rho v_z)}{\partial t}+\frac{1}{r}\frac{\partial (r\rho v_r v_z)}{\partial r}+\frac{1}{r}\frac{\partial (\rho {v_\theta}^2)}{\partial \theta}+\frac{\partial (\rho v_\theta v_z)}{\partial z}$$

$$=-\frac{\partial p}{\partial z}+\frac{1}{r}\frac{\partial r\sigma_{zr}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{z\theta}}{\partial \theta}+\frac{\partial \sigma_{zz}}{\partial z}+\rho f_z \quad (2.4)$$

The kinetic energy per unit mass of the fluid flow K is $\left(v_r^2+v_\theta^2+v_z^2\right)/2$. We can derive an equation for K from the continuity equation and the three momentum equations. The left-hand side of the r-component momentum equation (2.2) can be expanded to give

$$\rho\frac{\partial v_r}{\partial t}+\rho v_r\frac{\partial v_r}{\partial r}+\frac{\rho v_\theta}{r}\frac{\partial v_r}{\partial \theta}+\rho v_z\frac{\partial v_r}{\partial z}+v_r\frac{\partial \rho}{\partial t}+\frac{v_r}{r}\frac{\partial (r\rho v_r)}{\partial r}+\frac{v_r}{r}\frac{\partial (\rho v_\theta)}{\partial \theta}+v_r\frac{\partial (\rho v_z)}{\partial z}$$

$$=-\frac{\partial p}{\partial r}+\frac{\partial \sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta}+\frac{\partial \sigma_{rz}}{\partial z}+\frac{\sigma_{rr}}{r}-\frac{\sigma_{\theta\theta}}{r}+\rho f_r$$

From the continuity equation (2.1) the last four terms on the left-hand side of this equation are zero, so

$$\rho\frac{\partial v_r}{\partial t}+\rho v_r\frac{\partial v_r}{\partial r}+\frac{\rho v_\theta}{r}\frac{\partial v_r}{\partial \theta}+\rho v_z\frac{\partial v_r}{\partial z}$$

$$=-\frac{\partial p}{\partial r}+\frac{\partial \sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta}+\frac{\partial \sigma_{rz}}{\partial z}+\frac{\sigma_{rr}}{r}-\frac{\sigma_{\theta\theta}}{r}+\rho f_r$$

By multiplying through by v_r, we have

$$\rho\frac{\partial\left(\frac{v_r^2}{2}\right)}{\partial t}+\rho v_r\frac{\partial\left(\frac{v_r^2}{2}\right)}{\partial r}+\frac{\rho v_\theta}{r}\frac{\partial\left(\frac{v_r^2}{2}\right)}{\partial\theta}+\rho v_z\frac{\partial\left(\frac{v_r^2}{2}\right)}{\partial z}$$

$$=-v_r\frac{\partial p}{\partial r}+v_r\left[\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta}+\frac{\partial\sigma_{rz}}{\partial z}+\frac{\sigma_{rr}}{r}-\frac{\sigma_{\theta\theta}}{r}\right]+\rho v_r f_r \qquad (2.5)$$

Similarly, for the θ- and z-component momentum equations:

$$\rho\frac{\partial\left(\frac{v_\theta^2}{2}\right)}{\partial t}+\rho v_r\frac{\partial\left(\frac{v_\theta^2}{2}\right)}{\partial x}+\frac{\rho v_\theta}{r}\frac{\partial\left(\frac{v_\theta^2}{2}\right)}{\partial y}+\rho v_z\frac{\partial\left(\frac{v_\theta^2}{2}\right)}{\partial z}$$

$$=-\frac{v_\theta}{r}\frac{\partial p}{\partial\theta}+v_\theta\left[\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+\frac{\sigma_{\theta r}}{r}+\frac{\partial\sigma_{\theta r}}{\partial r}+\frac{\partial\sigma_{\theta z}}{\partial z}+\frac{\sigma_{r\theta}}{r}\right]+\rho v_\theta f_\theta \qquad (2.6)$$

$$\rho\frac{\partial\left(\frac{v_z^2}{2}\right)}{\partial t}+\rho v_r\frac{\partial\left(\frac{v_z^2}{2}\right)}{\partial x}+\frac{\rho v_\theta}{r}\frac{\partial\left(\frac{v_z^2}{2}\right)}{\partial y}+\rho v_z\frac{\partial\left(\frac{v_z^2}{2}\right)}{\partial z}$$

$$=-v_z\frac{\partial p}{\partial z}+v_z\left[\frac{1}{r}\frac{\partial r\sigma_{zr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{z\theta}}{\partial\theta}+\frac{\partial\sigma_{zz}}{\partial z}\right]+\rho v_z f_z \qquad (2.7)$$

By summing (2.5), (2.6) and (2.7), we obtain

$$\rho\frac{\partial K}{\partial t}+\rho v_r\frac{\partial K}{\partial r}+\frac{\rho v_\theta}{r}\frac{\partial K}{\partial\theta}+\rho v_z\frac{\partial K}{\partial z}$$

$$=-\left[v_r\frac{\partial p}{\partial r}+\frac{v_\theta}{r}\frac{\partial p}{\partial\theta}+v_z\frac{\partial p}{\partial z}\right]$$

$$+v_r\left[\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta}+\frac{\partial\sigma_{rz}}{\partial z}+\frac{\sigma_{rr}}{r}-\frac{\sigma_{\theta\theta}}{r}\right]+v_\theta\left[\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+\frac{\sigma_{\theta r}}{r}+\frac{\partial\sigma_{\theta r}}{\partial r}+\frac{\partial\sigma_{\theta z}}{\partial z}+\frac{\sigma_{r\theta}}{r}\right]$$

$$+v_z\left[\frac{1}{r}\frac{\partial r\sigma_{zr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{z\theta}}{\partial\theta}+\frac{\partial\sigma_{zz}}{\partial z}\right]$$

$$\rho v_r f_r+\rho v_\theta f_\theta+\rho v_z f_z \quad [\mathrm{J\ kg^{-1}\ s^{-1}}] \qquad (2.8)$$

Multiplying the continuity equation (2.1) by K gives

$$K\frac{\partial\rho}{\partial t}+\frac{K}{r}\frac{\partial(r\rho v_r)}{\partial r}+\frac{K}{r}\frac{\partial(\rho v_\theta)}{\partial\theta}+K\frac{\partial(\rho v_z)}{\partial z}=0$$

and adding this equation to the left-hand side of (2.8) gives

$$\frac{\partial\rho K}{\partial t}+\frac{1}{r}\frac{\partial(r\rho v_r K)}{\partial r}+\frac{1}{r}\frac{\partial(\rho v_\theta K)}{\partial\theta}+\rho v_z\frac{\partial(\rho v_z K)}{\partial z}$$

$$=-\left[v_r\frac{\partial p}{\partial r}+\frac{v_\theta}{r}\frac{\partial p}{\partial\theta}+v_z\frac{\partial p}{\partial z}\right]$$

$$+v_r\left[\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta}+\frac{\partial\sigma_{rz}}{\partial z}+\frac{\sigma_{rr}}{r}-\frac{\sigma_{\theta\theta}}{r}\right]+v_\theta\left[\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+\frac{\sigma_{\theta r}}{r}+\frac{\partial\sigma_{\theta r}}{\partial r}+\frac{\partial\sigma_{\theta z}}{\partial z}+\frac{\sigma_{r\theta}}{r}\right]$$

$$+v_z\left[\frac{1}{r}\frac{\partial r\sigma_{zr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{z\theta}}{\partial\theta}+\frac{\partial\sigma_{zz}}{\partial z}\right]$$

$$\rho v_r f_r+\rho v_\theta f_\theta+\rho v_z f_z \quad [\mathrm{J\ kg^{-1}\ s^{-1}}] \qquad (2.9)$$

which is the equation for the conservation of kinetic energy K in cylindrical coordinates.

3 References

The following reports may be downloaded from
https://atkinsonscience.co.uk/Downloads/FluidDynamics.aspx

1. K. N. Atkinson, *Equations of Fluid Flow, Continuity Equation in Cartesian and Cylindrical Coordinates, Theory Guide*, Atkinson Science Limited, 2020.
2. K. N. Atkinson, *Equations of Fluid Flow, Momentum Equations in Cartesian Coordinates, Theory Guide*, Atkinson Science Limited, 2020.
3. K. N. Atkinson, *Equations of Fluid Flow, Momentum Equations in Cylindrical Coordinates, Theory Guide*, Atkinson Science Limited, 2020.